EXPONENTS OF SOME N-COMPACT SPACES

BY

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ABSTRACT

For any topological space T, S. Mrówka has defined Exp (T) to be the smallest cardinal κ (if any such cardinals exist) such that T can be embedded as a closed subset of the product N^κ of κ copies of N (the discrete space of cardinality $\aleph_0$). We prove that for Q, the space of the rationals with the inherited topology, Exp (Q) is equal to a certain covering number, and we show that by modifying some earlier work of ours it can be seen that it is consistent with the usual axioms of set theory including the choice that this number equal any uncountable regular cardinal less than or equal to $2^{\aleph_0}$. Mrówka has also defined and studied the class $\mathcal{M} = \{\kappa: \text{Exp}\,(N_\kappa) = \kappa\}$ where N_κ is the discrete space of cardinality κ. It is known that the first cardinal not in $\mathcal{M}$ must not only be inaccessible but cannot even belong to any of the first ω Mahlo classes. However, it is not known whether every cardinal below $2^{\aleph_0}$ is contained in $\mathcal{M}$. We prove that if there exists a maximal family of almost-disjoint subsets of N of cardinality κ, then $\kappa \in \mathcal{M}$, and we then use earlier work to prove that if it is consistent that there exist cardinals which are not in the first ω Mahlo classes, then it is consistent that there exist such cardinals below $2^{\aleph_0}$ and that $\mathcal{M}$ nevertheless contain all cardinals no greater than $2^{\aleph_0}$. Finally, we consider the relationship between $\mathcal{M}$ and certain "large cardinals", and we prove, for example, that if μ is any normal measure on a measurable cardinal, then $\mu(\mathcal{M}) = 0$.

1. Introduction and notation

Let N be the set of natural numbers, let N be the topological space obtained by putting the discrete topology on N, and for any cardinal κ let N^κ be the space obtained by taking a product of κ copies of N with the usual product topology. S. Mrówka [8] had defined a topological space T to be N-compact iff there exists a cardinal κ such that T can be embedded as a closed subset of N^κ, and for such spaces T, he has defined Exp(T) to be the least cardinal κ for which the embedding exists.

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In particular, let Q be the set of rational numbers, and let $\mathbf{Q}$ be Q with the inherited topology. Mrówka [8, p. 184–185] mentions that $\mathbf{Q}$ is N-compact and that

$$\aleph_0 < \operatorname{Exp}(\mathbf{Q}) \leqq 2^{\aleph_0}.$$

In this paper we shall show that this is close to the best possible. That is we shall show that Exp$(\mathbf{Q})$ is equal to a certain covering number which we shall define and which will be related to the number of nowhere dense sets needed to cover the real line. Then, using a slight modification of an earlier construction of ours [3], we shall prove that it is consistent that this covering number be any regular uncountable cardinal no greater than $2^{\aleph_0}$.

We shall also consider a similar problem p oposed by Mrówka [7]. For any cardinal κ let N_κ be the discrete space of cardinality κ. Then Mrówka has defined the class

$$\mathscr{M} = \{\kappa : \operatorname{Exp}(N_\kappa) = \kappa\}$$

and has asked about its extent. Mrówka has noted that $\mathscr{M}$ contains only cardinals below the first measurable cardinal and that any cardinal in this range which is the cardinality of the power set of some cardinal is in $\mathscr{M}$. Mycielski [11] has proven that the first cardinal not in $\mathscr{M}$ (if there exists such a cardinal) must be inaccessible, and Mrówka [10] has extended this to show that the first such cardinal cannot be in any of the first ω Mahlo classes of inaccessible cardinals. (A cardinal κ is called inaccessible iff it is a limit cardinal but is not the union of fewer than κ smaller cardinals; we shall define Mahlo classes in §3. It is known to be consistent with the axioms of set theory that there do not exist any uncountable inaccessible cardinals.) We shall extend this by proving that, if it is consistent that there exist such Mahlo cardinals, then it is also consistent that there exist such cardinals in $\mathscr{M}$ which are strictly below the continuum. We shall do this by showing first that it is possible to construct the desired embedding of N_κ into N^κ with the aid of a maximal family of cardinality κ of almost-disjoint (having finite intersection) subsets of N, and then referring to some earlier work of ours [2] on the consistency of the existence of such families.

We shall conclude with some brief remarks vis-à-vis $\mathscr{M}$ and "large" cardinals. We shall note that the original interpretation [7] of $\mathscr{M}$ as the class of "strongly" nonmeasurable cardinals is correct in that if there exists a measurable cardinal, then $\mathscr{M}$ is, in a certain sense, a very small subset of the cardinals below this measurable cardinal. We shall do this by characterizing those strongly inaccessible cardinals which are in $\mathcal{M}$.

The major proofs in this paper will be combinatorial or topological in nature and will be essentially self-contained. The theorems involved will tell us that the existence of certain desired embeddings will be implied by, or equivalent to, the existence of certain families. We will refer to earlier work of ours and others to obtain, either directly or with only slight modifications, existence and independence results concerning the families and therefore the embeddings in question.

We begin with some notation. As usual we shall assume the axiom of choice throughout, and we shall identify cardinals as initial ordinals.

For any sets A and B and any function f we denote the set of functions from A into B by ${}^{A}B$, the set $\{a \in A : a \notin B\}$ by $A - B$, and the set $\{f(a) : a \in A\}$ by $f[A]$. As we already mentioned, we shall use N and Q to denote the sets of natural numbers and rational numbers respectively, and we shall use R to denote the set of all real numbers and I to denote $R - Q$. We shall also use $\boldsymbol{R}$ and $\boldsymbol{Q}$ to denote R and Q with the usual topologies, and we shall use $\boldsymbol{c}$ to denote $2^{\aleph_0}$.

Finally, because most of the consistency proofs in the literature are carried out with respect to this theory, we shall work within Zermelo-Fraenkel set theory including choice, and we shall denote this theory by ZFC. A brief description of this theory can be found in [1, Chapter 2].

2. Covering families and Exp($\boldsymbol{Q}$)

As we shall be interested in closed subsets of I, we define a family

$$\mathcal{F} = \{F \subseteq I : F \text{ is closed in } \boldsymbol{R}\}$$

which will remain fixed throughout this section. Next, we define a family $\mathcal{G} \subseteq \mathcal{F}$ to **cover** I iff $\cup \mathcal{G} = I$, and we define such a family to **strongly cover** I iff for every irrational number r there exists a set $G \in \mathcal{G}$ such that r is a two-sided limit point in G. We then define the **covering number** of I (which we shall denote by $\mathrm{Cov}(I)$) to be the least cardinal κ such that there exists a family $\mathcal{G} \subseteq \mathcal{F}$ of cardinality κ which covers I. We shall need:

LEMMA. *If κ is the covering number of I, then there exists a family $\mathcal{H} \subset \mathcal{F}$ of cardinality κ which strongly covers I.*

PROOF. Let $\mathcal{G} \subseteq \mathcal{F}$ be a family of cardinality κ which covers I and for each $G \in \mathcal{G}$ define

$$C_G = \{r \in G : r \text{ is not a two-sided limit point of } G\}.$$

It is well known that C_G must be at most countable. Thus we may choose a countable family $\mathscr{C}_G \subseteq \mathscr{F}$ such that for each $r \in C_G$ there exists a $D \in \mathscr{C}_G$ in which r is a two-sided limit. Now let

$$\mathscr{H} = \mathscr{S} \cup \bigcup\{\mathscr{C}_G : G \in \mathscr{G}\}. \qquad \blacksquare$$

Using these notions, we may state and prove the main theorem of this section.

THEOREM 1.† *$\mathrm{Exp}(Q)$ is equal to the covering number of I.*

PROOF. We first prove that $\mathrm{Cov}(I) \leqq \mathrm{Exp}(Q)$. Thus let λ be any cardinal such that there exists an embedding Φ of Q into N^λ whose range, which we denote by C, is closed. We may think of N^λ as having ${}^\lambda N$ as its set of elements, and for each ordinal $\alpha < \lambda$ we let $\phi_\alpha \in {}^Q N$ be the continuous projection function of α defined by

$$\phi_\alpha(q) = \Phi(q)(\alpha) \qquad \text{for all } q \in Q.$$

We then extend each ϕ to a continuous function ϕ^*_α defined on an open subset of $\boldsymbol{R}$ by setting

$$\phi^*_\alpha(r) = n \text{ iff } \exists p, q \in Q[p < r < q \wedge \forall s \in Q(p \leqq s \leqq q \rightarrow \phi_\alpha(s) = n)].$$

Because the domain of each ϕ^*_α is open and contains Q, each set

$$F_\alpha = I - \mathrm{dm}(\phi^*_\alpha) \in \mathscr{F}.$$

Therefore, to complete this part of our proof it is sufficient to prove that $\{F_\alpha : \alpha < \lambda\}$ covers I. But suppose otherwise, and let r be any member of $I - \bigcup_{\alpha<\lambda} F_\alpha$. Then r must belong to the domain of each ϕ^*_α, so we may define a point $f \in {}^\lambda N$ by setting

$$f(\alpha) = \phi^*_\alpha(r) \qquad \text{for all } \alpha < \lambda.$$

† S. Mrówka has noted in a private communication that this theorem may be proven from his work by using topological properties of certain compactifications. With his permission, we present his proof. "Let $\boldsymbol{R}^*$ be the extended reals $[-\infty, +\infty]$. Then Cov (I) is the smallest cardinal κ such that $\boldsymbol{R}^* - \boldsymbol{Q}$ is the union of κ *compact* sets. On the other hand, by (3.3) and (4.1) in [7], we have $\mathrm{Exp}(\boldsymbol{Q}) = \mathrm{def}_N(\boldsymbol{Q}) =$ the smallest cardinal κ such that $\beta_D \boldsymbol{Q} - \boldsymbol{Q}$ is the union of κ closed G_δ — subsets of $\beta_D\boldsymbol{Q}$. Now, $\beta\boldsymbol{Q}$ is 0-dimensional, hence $\beta_D\boldsymbol{Q} = \beta\boldsymbol{Q}$ [7, p. 598]. Furthermore, $\boldsymbol{R}^*$ is a compactification of Q (such that every compact subset of $\boldsymbol{R}^* - \boldsymbol{Q}$ is a closed G_δ-subset of $\boldsymbol{R}^*$); consequently, considering the canonical map of $\beta\boldsymbol{Q}$ onto $\boldsymbol{R}^*$, we obtain the required map immediately." Our proof will be more combinatorial and will be essentially self-contained.

We shall prove that f, while not a member of $\Phi[Q]$, is a member of its closure, thus contradicting our hypothesis that $\Phi[Q]$ is closed. We shall need to know if G is any open set in ${}^{\lambda}N$ containing f, then

$$(*) \qquad \exists p, q \in Q[p < r < q \wedge \forall s \in Q(p \leqq s \leqq q \rightarrow \Phi(s) \in G)].$$

To prove this we recall that by the definition of the product topology, there exists a finite set $A \subseteq \lambda$ such that

$$\{g \in {}^{\lambda}N : \alpha \in A \rightarrow g(\alpha) = f(\alpha)\} \subseteq G.$$

From the definition of the ϕ_α^* there must exist for each $\alpha \in A$ rationals $p_\alpha < r < q_\alpha$ such that

$$p_\alpha \leqq s \leqq q_\alpha \rightarrow \phi_\alpha(s) = \phi_\alpha^*(r) \qquad \text{for all } s \in Q.$$

Hence, if we set

$$p = \max(\{p_\alpha : \alpha \in A\}) \text{ and } q = \min(\{q_\alpha : \alpha \in A\},$$

then we have $p < r < q$ and

$$p \leqq s \leqq q \rightarrow \phi_\alpha(s) = \phi_\alpha^*(r) = f(\alpha) \qquad \text{for all } \alpha \in A \text{ and } s \in Q.$$

But this, in turn, implies that for any $s \in Q$ we have

$$p \leqq s \leqq q \rightarrow \Phi(s) \in G.$$

This tells us immediately that f is an accumulation point of $\Phi[Q]$. Now let s be any point in Q and let

$$F = \{t \in Q : |t - r| \leqq |s - r|/2\}.$$

Because Φ is an embedding and F is a closed (in Q) subset of Q not containing s, there must be an open set G in N^λ which contains $\Phi(s)$ and is disjoint from $\Phi[F]$. But by (*), f must be an accumulation point of $\Phi[F]$, so f cannot equal $\Phi(s)$. Therefore, since s was an arbitrary point in Q, f cannot belong to $\Phi[Q]$.

To prove that $\mathrm{Exp}(Q) \leqq \mathrm{Cov}(I)$, we shall construct an embedding Φ by constructing the necessary ϕ_α. We begin by setting $\kappa = \mathrm{Cov}(I)$ and choosing a family $\mathscr{H} \subseteq \mathscr{F}$ of cardinality κ which strongly covers I. For convenience, we index $\mathscr{H}$ with $\kappa - \omega$, i.e. we set

$$\mathscr{H} = \{H_\alpha : \omega \leqq \alpha < \kappa\}.$$

For finite ordinals n we define our ϕ_n in such a way as to "divide" Q into smaller and smaller intervals. Since, for the purposes of continuity, our division points must be at irrationals, we shall define them in terms of the irrational π. Thus for any finite ordinal n and any $q \in Q$ we set

$$\phi_n(q) = \begin{cases} 2 \cdot \min(\{m \in N : q < \pi + m/(n+1)\}) & \text{for } \pi < q, \\ -1 + 2 \cdot \min\{m \in N : \pi < q + m/(n+1)\}) & \text{otherwise.} \end{cases}$$

In defining the remaining ϕ_α, we shall use the H_α. Since for every infinite ordinal $\alpha < \kappa$ the set H_α is a closed subset of R, its complement in R can be expressed as the union of a countable family

$$\mathscr{J}^\alpha = \{J_n^\alpha : n \in N\}$$

of disjoint open intervals. Using such a decomposition, we define

$$\phi_\alpha(q) = n \leftrightarrow q \in J_n^\alpha \qquad \text{for all } q \in Q.$$

Because all of the intervals used in defining the ϕ_α have irrational endpoints, it is easily seen that all of the ϕ_α are continuous and thus so is the function Φ from $\boldsymbol{Q}$ into N^κ which they generate. From a theorem of Mrówka [8, Th. 2.1.c] it follows immediately that to prove that Φ is an embedding, it is sufficient to prove that for every point $q \in \boldsymbol{Q}$ and every closed (in $\boldsymbol{Q}$) set F not containing q, there is an α such that $\phi_\alpha(q) \notin \phi_\alpha[F]$. But this is obvious from the construction of $\{\phi_\alpha : \alpha < \omega\}$.

We complete our proof by showing that Φ is a closed embedding, i.e. that for each $f \in {}^\kappa N$ not in the range of Φ, there exists an open set G in N^κ which contains f and is disjoint from the range of Φ. So choose any such f, and let

$$\mathscr{I} = \{\mathrm{cl}(\phi_n^{-1}(f(n)) : n < \omega\}$$

where "cl" is the closure operator taken with respect to $\boldsymbol{R}$. Because the intervals in $\mathscr{I}$ become arbitrarily small, the set $\cap\mathscr{I}$ can contain at most one point. We consider three cases.

Case 1. The set $\cap\mathscr{I}$ is empty. Then because $\mathscr{I}$ is a set of compact intervals there must exist $m, n < \omega$ such that

$$\phi_m^{-1}(f(m)) \cap \phi_n^{-1}(f(n)) = \varnothing.$$

Thus the open set

$$G = \{g \in {}^\kappa N : g(m) = f(m) \wedge g(n) = f(n)\}$$

contains f and is disjoint from $\Phi[Q]$.

Case 2. $\cap\mathscr{I}$ contains a single point q which is rational. Since f is not in the range of Φ, there is some $\alpha < \kappa$ such that

$$f(\alpha) \neq \Phi(b)(\alpha) = \phi_\alpha(q).$$

But each ϕ is continuous so there must exist an $n < \omega$ such that for any $p \in Q$ we have

$$|p - q| < 1/(n+1) \rightarrow \phi_\alpha(p) = \phi_\alpha(q).$$

Thus the open set we are looking for is

$$G = \{g \in {}^\kappa N : g(n) = f(n) \wedge g(\alpha) = f(\alpha)\}.$$

Case 3. $\cap \mathscr{J}$ contains a single point r which is irrational. By our assumption on $\mathscr{H}$ there is an α such that r is a two-sided limit point in H_α. Thus r cannot be a limit point of $J^\alpha_{f(\alpha)}$. In particular, there must exist an $n < \omega$ such that

$$J^\alpha_{f(\alpha)} \cap \{p \in Q : |p - r| < 1/(n+1)\} = \emptyset.$$

Then again we set

$$G = \{g \in {}^\kappa N : g(n) = f(n) \wedge g(\alpha) = f(\alpha)\}. \qquad \blacksquare$$

Although $\mathrm{Cov}(I)$ does not appear to be any easier to calculate than $\mathrm{Exp}(Q)$ itself, it is, in fact, easier to deal with because it is closely related in definition to the number of nowhere dense sets needed to cover $\boldsymbol{R}$. Thus if we call this latter number $\mathrm{Cov}(R)$, then we have immediately

$$\aleph_0 < \mathrm{Cov}(R) \leqq \aleph_0 + \mathrm{Cov}(I) = \mathrm{Cov}(I) = \mathrm{Exp}(Q) \leqq c.$$

But it is known that in Cohen's models [1, 16] and in models satisfying Martin's axiom [6], we have $\mathrm{Cov}(R) = c$ so in these models $\mathrm{Exp}(Q) = c$. Similarly, it is known that in Solovay's models where random reals are added [14] we have $\mathrm{Cov}(R) = \aleph_1$, and it is not hard to extend the proof to obtain $\mathrm{Cov}(I) = \aleph_1$. However, by extending a construction of the author's we can obtain even more.

THEOREM 2. *It is consistent with* ZFC *that* $\mathrm{Exp}(Q)$ *be any uncountable regular cardinal less than or equal to* c.

PROOF. In [3] we proved a similar theorem about $\mathrm{Cov}(R)$ and here it is only necessary to note that a very slight modification of that construction yields our present theorem. In particular, if we look at the construction in [3], we see that the nowhere dense sets we construct are complements of open sets and contain only those real numbers which are explicitly forced to be there by the presence of terms denoting them in the conditions. Thus it will be sufficient to exclude all terms denoting rationals from our conditions. Perhaps the simplest way to do this would be to add to 3.1.2.3* the additional clause (c):

$$p' \Vdash t \notin Q .$$

We leave the details to the interested reader. ∎

Since this paper was written, the author and S. Mrówka have been able to prove that the cofinality of $\mathrm{Exp}(Q)$ must be uncountable and that, subject only to this restriction, it is consistent that $\mathrm{Exp}(Q)$ be any cardinal less than or equal to c. The proof will appear elsewhere [4].

3. Almost-disjoint families and the class $\mathscr{M}$

As we noted in §1, Mycielski [11] has proven that the first cardinal not in $\mathscr{M}$ must be inaccessible and that it is, therefore, consistent that $\mathscr{M}$ contain all cardinals. Furthermore, suppose we put the order topology on the ordinals and for any set S of ordinals, we use $\bar{S}$ to denote its closure. Then, following Mahlo [5], given any class C of cardinals we define a new class

$$M(C) = \{\alpha : \exists A \subseteq C(A \cup \{\alpha\} = \bar{A})\},$$

and for any ordinal α we define $M_\alpha(C)$ by setting:

$$M_0(C) = C,$$

$$M_{\beta+1}(C) = M(M_\beta(C)),$$

and

$$M_\lambda(C) = \bigcup_{\beta<\lambda} M_\beta(C) \qquad \text{for limit ordinals } \lambda.$$

Next, for the remainder of this section let $\mathfrak{U}$ be the class of all cardinals which are not weakly inaccessible. Then while Mycielski's theorem tells us only that the first cardinal not in $\mathscr{M}$ cannot be in $\mathfrak{U}$, Mrówka [10] has recently proven that this first cardinal cannot be in $M_\omega(\mathfrak{U})$. However, as Mrówka notes [10, p. 195], this does not necessarily give us complete information even about the cardinals below the continuum. We shall prove that it is consistent with ZFC that all cardinals up to the continuum belong to $\mathscr{M}$ even if for some $\alpha > \omega$ some of these cardinals are not in $M_\alpha(\mathfrak{U})$.

We begin by defining a family of subsets of N to be an **almost-disjoint family** iff it is infinite, each of its members is infinite, and the intersection of any two distinct members of it is finite. A **maximal** almost-disjoint family is then defined to be an almost-disjoint family which is not properly contained in any other almost-disjoint family. Using such families, we can mimic a construction due to Mrówka [9, Th. 2] to obtain:

THEORFM 3. *If there exists a maximal almost-disjoint family of cardinality* κ, *then* $\kappa \in \mathscr{M}$.

PROOF. It is clearly sufficient to prove that N^κ contains a closed discrete subset D of cardinality κ. Let $\mathscr{F}$ be a maximal almost-disjoint family of cardinality κ, and, as in the proof of Theorem 1, index it with $\kappa - \omega$, i.e. set

$$\mathscr{F} = \{F_\alpha \subseteq N : \omega \leqq \alpha \leqq \kappa\}.$$

We shall again think of N^κ as having ${}^\kappa N$ for its set of elements, and we shall need two auxiliary functions Ψ and θ. For Ψ we may choose any bijection from the set of all finite subsets of N onto $N-\{1\}$, while θ is to be the function defined for each infinite $A \subseteq N$ and each $n \in N$ by

$$\theta(A,n) = \min(A - \{\theta(A,m) : m < n\}).$$

(Thus $\theta(A,n)$ is the nth smallest member of A.) Using these, we define for each $F \in \mathscr{F}$ a unique point $f_F \in {}^\kappa N$ by setting

$$f_F(\alpha) = \begin{cases} \theta(F, \alpha+1) & \text{for } \alpha < \omega, \\ 1 & \text{for } F = F_\alpha, \\ \Psi(F \cap F_\alpha & \text{otherwise.} \end{cases}$$

We shall prove that the set

$$D = \{f_F : F \in \mathscr{G}\}$$

is the required discrete closed subset of N^κ of cardinality κ.

We note immediately that D has cardinality κ, so we need only find for each $f \in {}^\kappa N$ an open set G_f containing f such that the set $G_f \cap D$ is finite. Thus choose any $f \in {}^\kappa N$. We note that every function $g \in D$ is completely determined by $g[\omega]$, and we shall use this to define the set G_f. We shall consider several cases.

Case 1. f is not strictly increasing over ω. Then for some $m < n < \omega$ we have $f(n) \leqq f(m)$, and we may set

$$G_f = \{g \in {}^\kappa N : g(m) = f(m) \wedge g(n) = f(n)\}.$$

Case 2. For some infinite ordinal α we have $f(\alpha) = 1$. Since the only member of D to have this property is f_{F_α}, we may set

$$G_f = \{g \in {}^\kappa N : g(\alpha) = 1\}.$$

Case 3. f is strictlyi ncreasing over ω and does not take on the value 1 at any infinite ordinal. Thus $f[\omega]$ must be infinite, and, because $\mathscr{F}$ was chosen to be a *maximal* almost-disjoint family, there must exist an infinite ordinal α such that $f[\omega] \cap F_\alpha$ is infinite. Choose any point

$$n \in (F_\alpha \cap f[\omega] - \Psi^{-1}(f(\alpha))$$

(this can be done because $\Psi^{-1}(f(\alpha))$ is finite) and set

$$G_f = \{g \in {}^\kappa N : g(\alpha) = f(\alpha) \wedge g(f^{-1}(n)) = n\}. \qquad \blacksquare$$

Using this we may prove:

THEOREM 4. *If for some ordinal α it is consistent with ZFC that there exist cardinals not in $M_\alpha(\mathfrak{U})$, then it is also consistent with ZFC that there exist such cardinals below the continuum and that $\mathcal{M}$ nevertheless contain all cardinals below the continuum.*

PROOF. Suppcse in some countable model of ZFC for some ordinal α we have a cardinal $\kappa \notin M_\alpha(\mathfrak{U})$. First, using Cohen's [1] original construction, we can obtain an extension of the model in which κ is less than the continuum. Then using a construction due to the present author [3, Th. 3.2], we can again extend the model in such a way as to insure that there exist maximal almost-disjoint families of all uncountable cardinalities below that of the continuum. In this final model κ will clearly be in $\mathcal{M}$, but we must check to see that it has not also become a member of $M_\alpha(\mathfrak{U})$. This, however, follows from the fact that each of the extensions used satisfies what is usually referred to as the "countable chain condition", and such extensions can be shown to preserve Mahlo classes. Finally, to avoid the use of countable models, the entire argument can be reframed in terms of Boolean models [15]. $\blacksquare$

We do not know whether or not it is a theorem of ZFC that every cardinal below c belongs to $\mathcal{M}$.† We do know that even with respect to uncountable cardinals below c, the converse of Theorem 3 is false. In particular, Solovay and Tennenbaun [15] have proven the relative consistency of Martins's axiom with the negation of the continuum hypothesis, and Martin and Solovay [6] have shown that Martin's axiom implies that every maximal almost-disjoint family has cardinality c. Thus, although by Mycielski's theorem $\aleph_1$ must be in $\mathcal{M}$, it is consistent that there exist no maximal almost-disjoint families of cardinality $\aleph_1$.

4. Large cardinals and the class $\mathcal{M}$

We conclude with some general remarks concerning the extent of $\mathcal{M}$. Let $\mathcal{V}$ be the class of all cardinals, and let $\mathcal{U}$ be the class of all cardinals which are

† It has since been shown that if it is consistent that there exist cardinals not in M, it is also consistent that there exist such cardinals less than $2^{\aleph_0}$.

below the first measurable cardinal. Then $\mathcal{M} \subseteq \mathcal{U}$, and Mycielski's result immediately implies the consistency of $\mathcal{M} = \mathcal{V}$. What is more surprising, however, is that $\mathcal{M} = \mathcal{U}$ implies $\mathcal{U} = \mathcal{V}$. Restated, the existence of a measurable cardinal κ implies the existence of a cardinal $\lambda < \kappa$ which is also not in $\mathcal{M}$. In fact, if κ is any measurable cardinal and μ is any normal measure (for the definition of normal measure see [12]), then $\mu(\mathcal{M}) = 0$. Thus, the existence of a measurable cardinal implies that, in a sense, $\mathcal{M}$ is a very small subset of $\mathcal{U}$.

The above is best proven using the notion of weakly compact cardinals. Weakly compact were originally defined in terms of infinitary languages and are precisely those cardinals which Mrówka [7, p. 604] defines as not being "strongly incompact". Silver [13], on the other hand, has proven that these cardinals can be characterized as the class of cardinals κ which are strongly inaccessible and which have the further property that every tree of cardinality κ which has fewer than κ elements at each level has a branch of cardinality κ. Using this, it is easy to prove that:

THEOREM 5. *If κ is any strongly inaccessible cardinal in $\mathcal{U}$, then $\kappa \in \mathcal{M}$ iff κ is not weakly compact.* ■

The fact that weakly compact cardinals cannot belong to $\mathcal{M}$ has long been known [7, p. 604], and a method of proving this can be found in [9, p. 708]. Furthermore it has also long been known that if κ is any measurable cardinal and μ is any normal measure on κ, then

$$\mu(\{\lambda < \kappa : \lambda \text{ is a weakly compact cardinal}\}) = 1.$$

Thus, as we stated earlier, we obtain

THEOREM 6. *If $\mathcal{U} \neq \mathcal{V}$ then:*

1. *$\mathcal{M} \neq \mathcal{U}$, and*

2. *If κ is any measurable cardinal and μ is any normal measure on κ, then $\mu(\mathcal{M}) = 0$.* ■

Mrówka has pointed out in a private communication that the remaining half of Theorem 5 can also be proven directly from earlier results. He points out that a similar theorem holds for a class $\mathcal{M}^*$ which he has defined and studied [7, 9] (the theorem follows immediately from his remarks in [9, p. 706]), and then goes on to prove that strongly inaccessible cardinals in $\mathcal{U}$ are in $\mathcal{M}$ iff they are in $\mathcal{M}^*$. For the convenience of the reader, and again with his permission, we present this proof.

The class $\mathcal{M}^*$ is defined to be the set of all cardinals κ such that there exists an embedding of the discrete space N_κ of cardinality κ onto a closed subspace of a product space $\Pi_{\alpha<\kappa} N_\alpha$ where each N_α is a discrete space of cardinality $|\alpha| < \kappa$. Clearly $\mathcal{M} \subseteq \mathcal{M}^*$, and Mrówka shows:

THEOREM 7 (Mrówka). *If κ is any cardinal in $\mathcal{U}$* such that $\lambda < \kappa \to 2^\lambda \leqq \kappa$, *then $\kappa \in \mathcal{M}^* \to \kappa \in \mathcal{M}$.*

PROOF. Since $\kappa \in \mathcal{M}^*$, we have a closed embedding of the space N_κ into $\Pi_{\alpha<\kappa} N_\alpha$. But κ and therefore each $\alpha < \kappa$ belong to $\mathcal{U}$, so we know that each N_α can be embedded as a closed suspace of N^{2^α}. Hence N_κ can be embedded as a closed subspace of $N^{\Sigma_{\alpha<\kappa}2^\alpha}$. But our original condition on κ implies that

$$\sum_{\alpha<\kappa} 2^\alpha \leqq \sum_{\alpha<\kappa} \kappa = \kappa^2 = \kappa, \text{ so we are done.}$$

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